# Emergence is relative

Taylor Hinchliffe

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**Abstract**

Emergent phenomena appear when interacting components give rise to novel behaviors not seen at the scale of their individual parts, resulting in new behaviors and complex systems that are very difficult to predict and understand through analysis of their individual components alone. More specifically, a population of microsystems unpredictably interacts to produce a new macrosystem. However, the unpredictable element of emergent systems can be removed by changing the observer perspective in a manner that adds information. This can be performed in space, as with an expansion of the observer's 3D frame of reference as if 'zooming out', in time, such as through increasing the observation period to capture the emergence of repetitive system behaviors, and in a 'five-dimensional' region of space, time and probability, through assigning observation probability ranges to the observer. Therefore, through the use of simplified mathematical models, the unpredictability of emergent phenomena is shown to be relative and based on observer perspective in coordinate space for both classical and chaotic systems. This emphasizes the importance of studying emergent phenomena, because with the addition of information, higher dimensions of perspective, and computational power, something unpredictable at one point in history may become predictable at a later time in history.

## Introduction

When a population of microsystems interacts to generate a novel and unpredictable macrosystem, an emergent behavior has appeared. Although definitions of emergence can vary among academic groups, mathematically, this is often thought of as high-energy, far-from-equilibrium systems that unpredictably self-organize to reduce chaotic energy flows (Grimes 2017, Rupe and Crutchfield 2024). Emergent phenomena span all known spatial and temporal scales, and have included systems and behaviors from the European Renaissance, to the emergence of multicellular life and consciousness, to large scale climate patterns, to modern large language AI models (Zizzi 2000, Hinojosa 2009, Weber 2010, Lovejoy and Schertzer 2018, de Zarzà, de Curtò et al. 2023). However, when system-relevant information is added to the observer perspective in the coordinate space, a once unpredictable system can become mathematically definable and predictable.

For example, in a classical system of equations, a set of n microsystems may have an initially definable set of descriptive formulas, while the resulting macrosystem may have yet another, distinct set of descriptive formulas. Yet if the observer modifies their coordinate frame of reference, including space, time and probability, interacting hidden systems may be encountered to either or both the microsystems and macrosystem. Through the analysis of hidden systems and their relationships to microsystems, and then through optionally studying their collective relationship to the macrosystem, a coherent function can be constructed for which predicts the macrosystem from its microsystems. Furthermore, it can be shown that analysis of hidden system elements and their relationships to microsystems can construct and therefore predict a macrosystem without prior knowledge of it.

Adding information to the observer perspective through modifying space, time and probability coordinates can also be extended to explain the appearance of order from simple chaotic systems. For example, it can be shown that a hidden system can influence values of a simple logistic map in a distance, time and probability-dependent manner, driving the transition of chaotic behavior to orderly, periodic behavior. The observer can expand their x, y, and z coordinate frame of reference to observe the hidden system and mathematically explain the transition from chaos to order, as well as increasing their observation period to capture the behavioral transition, and increasing a probability range for observing the transition.

An even more simplified mathematical representation of emergent phenomena might include microsystems A and B multiplied by a chaos-factor $\kappa$, such as $\kappa_A * A(t)$ and $\kappa_B * B(t)$. When a hidden 'emergence factor' $\kappa_E * E(t)$ is discovered by the observer, a collective system $\kappa_C * C(t)$ can be described as $\kappa_C * C(t) = \kappa_A * A(t) + \kappa_B * B(t) + \kappa_E * E(t)$, where $\kappa_C < \kappa_A + \kappa_B + \kappa_E$ , indicating a lower degree of chaos than the sum of the initial individual components. The appearance of the 'emergence factor', which could describe a new interaction or variables that lie beyond the observer's frame of reference, can be described as a 'hidden system'. From this, it can be shown how changes in the coordinate space of the observer perspective can result in the appearance of new system behaviors. Since a variety of mathematical models have been used to model different emergent phenomena, and therefore, there is not a singular set of equations commonly associated with emergence, three different models of varying complexity will be used to demonstrate the general principle of adding information to the observer perspective to reveal system information. For simplicity, deterministic and periodic functions will initially be used, following by a simplified chaotic system in the form of a logistic map, as well as a novel general model composed of a Fourier series. It will be shown that the discovery of hidden system components can be determined based upon the observer frame of reference in 3D space, as well as observation time and the probability of observation.

## Expanding 3-dimensional frames of reference to observe emergent phenomena

In returning to the simple classical system as a starting point, let us first describe our microsystems, hidden systems, and macrosystem in $x, y, z$ coordinate space. We may also assume that the total macrosystem in 3D coordinate space — in particular, the hidden systems required to collectively define it — are outside of the bounds of the observer's coordinate frame of reference, and therefore, is initially hidden to the observer.

To ensure that the macrosystem's total size coordinates do not initially include the hidden systems, we can define the functions such that the hidden systems are initially outside the observer's frame of reference. The set of $n$ microsystems as a set of time-dependent functions, $\mathbf{M}_i(t) = (x_i(t), y_i(t), z_i(t)) \quad \text{for} \quad i = 1,2,\ldots, n$, can be defined as $\mathbf{M}_1(t) = (\sin(t), \cos(t), t)$ and $\mathbf{M}_2(t) = (\cos(t), \sin(t), t)$. The macrosystem function describes the collective behavior of the microsystems, $\mathbf{C}(t) = (X(t), Y(t), Z(t))$, which can be specified as $\mathbf{C}(t) = (t^2, \sin(t) + \cos(t), t)$. Hidden systems H are additional functions that become relevant when "zooming out" or expanding the observer's frame of reference in coordinate space. They are initially outside the observer's frame of reference: $\mathbf{H}_j(t) = (u_j(t), v_j(t), w_j(t)) \quad \text{for} \quad j = 1,2,\ldots, m$
And can be specified as $\mathbf{H}_1(t) = (t + 10, \exp(t), \exp(-t))$ and $\mathbf{H}_2(t) = (t - 10, \log(t + 1), \sqrt{t})$.
To ensure the hidden systems are initially outside the observer's perspective, we can set boundaries for the observer's initial frame of reference. Let the initial observer frame of reference be defined by the bounds $[x_{\min}, x_{\max}], [y_{\min}, y_{\max}]$, and $[z_{\min}, z_{\max}]$. Example bounds can be defined as: $x_{\min} = -5, x_{\max} = 5, y_{\min} = -2, y_{\max} = 2, z_{\min} = 0, z_{\max} = 5$. Initially, the observer's frame of reference can be represented as:

$$\mathbf{R} = \{(x, y, z) \mid x_{\min} \leq x \leq x_{\max}, y_{\min} \leq y \leq y_{\max}, z_{\min} \leq z \leq z_{\max}\}$$

To include the coordinates of the hidden systems, we expand the observer's frame of reference, which can be specified as: $x'_{\min} = -15, x'_{\max} = 15, y'_{\min} = -10, y'_{\max} = 10, z'_{\min} = -5$, $z'_{\max} = 10$. The expanded frame of reference can be represented as:

$$\mathbf{R}' = \{(x, y, z) \mid x'_{\min} \leq x \leq x'_{\max}, y'_{\min} \leq y \leq y'_{\max}, z'_{\min} \leq z \leq z'_{\max}\}$$

In summary, the initially observed conditions are
$\mathbf{M}_1(t) = (\sin(t), \cos(t), t), \quad \mathbf{M}_2(t) = (\cos(t), \sin(t), t)$ for microsystems, $\mathbf{C}(t) = (t^2, \sin(t) + \cos(t), t)$ for the macrosystem, $\mathbf{H}_1(t) = (t + 10, \exp(t), \exp(-t)), \quad \mathbf{H}_2(t) = (t - 10, \log(t + 1), \sqrt{t})$ for the

hidden systems, and $\mathbf{R} = \{(x, y, z) \mid -5 \leq x \leq 5, -2 \leq y \leq 2, 0 \leq z \leq 5\}$ for the initial frame of reference. The observer then expands their frame of reference to:

$$\mathbf{R}' = \{(x, y, z) \mid -15 \leq x \leq 25, -10 \leq y \leq 10, -5 \leq z \leq 30\}$$

Therefore, the hidden systems $\mathbf{H}_1(t)$ and $\mathbf{H}_2(t)$ are initially outside the observer's perspective. By expanding the observer's frame of reference, the hidden systems are included, demonstrating how the emergent macrosystem can be better understood with additional information by expanding the observer perspective in 3D coordinate space.

To briefly analyze how the hidden systems 3D coordinates are initially outside of the observer's frame of reference, one can first redefine the initial bounds of the observer's frame of reference: $x_{\text{min}} = -5, x_{\text{max}} = 5, y_{\text{min}} = -2, y_{\text{max}} = 2, z_{\text{min}} = 0, z_{\text{max}} = 5$. The initial frame of reference is therefore:

$$\mathbf{R} = \{(x, y, z) \mid -5 \leq x \leq 5, -2 \leq y \leq 2, 0 \leq z \leq 5\}$$

The hidden systems are then restated: $\mathbf{H}_1(t) = (t + 10, \exp(t), \exp(-t))$ and $\mathbf{H}_2(t) = (t - 10, \log(t + 1), \sqrt{t})$.

One may then analyze the coordinates of these hidden systems to see if they fall within the initial frame of reference. For the first hidden system, $\mathbf{H}_1(t) = (t + 10, \exp(t), \exp(-t))$, observing the X-coordinate at $t + 10$, for $t = 0, x = 10$ (which is outside the range $[-5,5]$), for $t = 1, x = 11$ (which is outside the range $[-5,5]$). Generally, $t + 10$ will always be greater than 10, thus outside the initial $x$-range. For the Y-coordinate, $\exp(t)$, for $t = 0, y = 1$(which is within the range $[-2,2]$), and for $t = 1, y = e \approx 2.718$ (which is outside the range $[-2,2]$). Again, $\exp(t)$ grows exponentially, so for $t > 0$, $y$ will be outside the initial $y$-range. For the Z-coordinate, $\exp(-t)$, for $t = 0, z = 1$ (which is within the range $[0,5]$), and for $t = 1, z = \frac{1}{e} \approx 0.368$ (which is within the range $[0,5]$). Therefore, $\exp(-t)$ decreases but remains within the initial $z$-range. Since the $x$-coordinate and $y$-coordinate are initially outside the observer's frame of reference, $\mathbf{H}_1(t)$ is outside the initial frame of reference.

To analyze the second hidden system, $\mathbf{H}_2(t) = (t - 10, \log(t + 1), \sqrt{t})$ at the X-coordinate through $t - 10$, for $t = 0, x = -10$ (which is outside the range $[-5,5]$), and for $t = 5, x = -5$ (which is at the boundary of the range $[-5,5]$). Therefore, $t - 10$ will be less than -5 for $t < 15$, thus outside the initial $x$-range. For the Y-coordinate: $\log(t + 1)$, for $t = 0, y = \log(1) = 0$ (which is within the range $[-2,2]$), and for $t = 6.389, y \approx 2$ (which is at the boundary of the range $[-2,2]$). Again, $\log(t + 1)$ grows slowly, but for $t > 6.389$, $y$ will be outside the initial $y$-range. Lastly, for the Z-coordinate $\sqrt{t}$, for $t = 0, z = 0$ (which is within the range $[0,5]$), for $t = 4, z = 2$ (which is within the range $[0,5]$), and for $t = 25, z = 5$ (which is at the boundary of the range $[0,5]$). Therefore, $\sqrt{t}$ will be within the range for $t \leq 25$. Since the $x$-coordinate is initially outside the observer's frame of reference, $\mathbf{H}_2(t)$ is also outside the initial frame of reference.

To include the coordinates of the hidden systems within the observer's frame of reference, we need to expand the frame of reference in the coordinate space: The expanded bounds are defined as $x'_{\text{min}} = -15, x'_{\text{max}} = 15, y'_{\text{min}} = -10, y'_{\text{max}} = 10, z'_{\text{min}} = -5, z'_{\text{max}} = 10$. The expanded frame of reference equation is given as

$$\mathbf{R}' = \{(x, y, z) \mid -15 \leq x \leq 30, -10 \leq y \leq 10, -5 \leq z \leq 30\}$$

Therefore, the initial frame of reference, defined by $\mathbf{R}$, which does not include the coordinates of the hidden systems. The hidden systems initially have $x, y,$ and $z$ coordinates outside the bounds of $\mathbf{R}$. However, the expanded frame of reference $\mathbf{R}'$ includes the coordinates of the hidden systems, demonstrating that these systems are initially outside the observer's perspective.

## Expanding temporal observation period to observe emergent phenomena

In addition to expanding the observer perspective in 3D coordinate space, we may also increase the observation period in time to capture additional details that lead to an understanding of an emergent system. To see this in a simplified, classical (non-chaotic) system, we can incorporate repetitive and cyclic behavior into the model, define periodic functions for the microsystems and hidden systems, and then connect these behaviors to the macrosystem. Time $t$ is added to the observer's frame of reference to show how the cyclic behavior is revealed with sufficient observational time. The observation time can initially be too short to see a complete periodic cycle, and therefore, yield insufficient information to understand a system that might appear emergent. When the observation time is increased so that a full periodic cycle can be observed, phenomena that appear unpredictable can then become explained.

As before, one can start by defining $n$ microsystems, but this time with periodic behavior, $\mathbf{M}_i(t) = (x_i(t), y_i(t), z_i(t)) \quad \text{for} \quad i = 1,2,\ldots, n$. An example specification could be $\mathbf{M}_1(t) = (\sin(t), \cos(t), t)$ and $\mathbf{M}_2(t) = (\cos(t), \sin(t), t)$. We can then define $m$ hidden systems with periodic behavior as $\mathbf{H}_j(t) = (u_j(t), v_j(t), w_j(t)) \quad \text{for} \quad j = 1,2,\ldots, m$, such as $\mathbf{H}_1(t) = (t + 10, \exp(\sin(t)), \exp(-\cos(t)))$ and $\mathbf{H}_2(t) = (t - 10, \log(\cos(t) + 2), \sqrt{\sin(t) + 1})$.
The macrosystem function then describes the collective behavior influenced by the periodic nature of microsystems and hidden systems: $\mathbf{C}(t) = (X(t), Y(t), Z(t))$.
Then, to connect the behaviors of the microsystems and hidden systems to the macrosystem through cyclic behavior, we can define the macrosystem as a function of both:

$$X(t) = f_1\left(\sum_{i=1}^{n} x_i(t), \sum_{j=1}^{m} u_j(t)\right)$$

$$Y(t) = f_2\left(\sum_{i=1}^{n} y_i(t), \sum_{j=1}^{m} v_j(t)\right)$$

$$Z(t) = f_3\left(\sum_{i=1}^{n} z_i(t), \sum_{j=1}^{m} w_j(t)\right)$$

Which can be specified, for example, as: $X(t) = \sin(t) + \cos(t) + \exp(\sin(t)) + (t + 10)$, $Y(t) = \cos(t) + \sin(t) + \exp(-\cos(t)) + \log(\cos(t) + 2)$ and $Z(t) = t + t + \sqrt{\sin(t) + 1} + \exp(-\cos(t))$. Then, one can add a dimension to the observer perspective, so that the observer's frame of reference is now in spacetime $(x, y, z, t)$. Let the initial bounds be set as $x_{\min} = -5, x_{\max} = 5$, $y_{\min} = -2, y_{\max} = 2, z_{\min} = 0, z_{\max} = 5, t_{\min} = 0, t_{\max} = \pi$ (less than one full cycle for $\sin(t)$). Therefore, the initial frame of reference can be stated as:

$$\mathbf{R} = \{(x, y, z, t) \mid -5 \leq x \leq 5, -2 \leq y \leq 2, 0 \leq z \leq 5, 0 \leq t \leq \pi\}$$

The inclusion of two full cycles into the observer perspective then appears as $t'_{\max} = 4\pi$, with the expanded frame of reference now becoming:

$$\mathbf{R}' = \{(x, y, z, t) \mid -15 \leq x \leq 30, -10 \leq y \leq 10, -5 \leq z \leq 30, 0 \leq t \leq 4\pi\}$$

Therefore, we start with the initial systems as follows: microsystems $\mathbf{M}_1(t) = (\sin(t), \cos(t), t), \quad \mathbf{M}_2(t) = (\cos(t), \sin(t), t)$, hidden systems $\mathbf{H}_1(t) = (t + 10, \exp(\sin(t)), \exp(-\cos(t))), \quad \mathbf{H}_2(t) = (t - 10, \log(\cos(t) + 2), \sqrt{\sin(t) + 1})$, and the macrosystem:

$$\mathbf{C}(t) = \left(\sin(t) + \cos(t) + \exp(\sin(t)) + (t + 10), \cos(t) + \sin(t) + \exp(-\cos(t)) + \log(\cos(t) + 2), t + t + \sqrt{\sin(t) + 1} + \exp(-\cos(t))\right)$$

While the the observer's initial frame of reference is arbitrarily stated as:

$$\mathbf{R} = \{(x, y, z, t) \mid -5 \leq x \leq 5, -2 \leq y \leq 2, 0 \leq z \leq 5, 0 \leq t \leq \pi\}$$

Such is then expanded to the following to include two full cycles:

$$\mathbf{R}' = \{(x, y, z, t) \mid -15 \leq x \leq 30, -10 \leq y \leq 10, -5 \leq z \leq 30, 0 \leq t \leq 4\pi\}$$

By incorporating repetitive/cyclic behaviors that arise from the microsystem and hidden systems, one can demonstrate how the observer's perspective in spacetime can influence the perception of these behaviors. By expanding the frame of reference in time, the previously hidden cyclic behaviors are revealed, connecting the microsystem and hidden system behaviors to the macrosystem. The $x$-component $X(t) = \sin(t) + \cos(t) + \exp(\sin(t)) + (t + 10)$ ranges from 11.55 to approximately 24.57 (now within the expanded $x$-range $[-15, 30]$). The $y$-component $Y(t) = \cos(t) + \sin(t) + \exp(-\cos(t)) + \log(\cos(t) + 2)$ ranges from about 0.37 to 7.72 (within the expanded $y$-range $[-10, 10]$). The $z$-component $Z(t) = 2t + \sqrt{\sin(t) + 1} + \exp(-\cos(t))$ ranges from 1.37 to approximately 26.5 (now within the expanded $z$-range $[-5, 30]$).

Therefore, by expanding the observer's frame of reference to include a longer time span, the initially hidden cyclic behaviors and the coordinates of the hidden systems fall within the expanded spacetime frame. This demonstrates how emergent properties and behaviors can become observable when the observer's frame of reference is sufficiently extended, connecting the microsystem and hidden system behaviors to the macrosystem.

In addition to 'hiding' the $x, y, z$ values of the hidden systems until a certain temporal value is reached, therefore revealing the full 3D system once the observer's frame of reference in time is expanded to include a sufficient time span to observe full cyclic behaviors, it is also possible to 'discover' a new function that describes the collective behavior arising from both microsystems and hidden systems, which can in turn be aggregated into a new collective system.

To illustrate this, one can once again define our microsystems, hidden systems, macrosystem and collective behavior function. $n$ microsystems with periodic behavior are defined as $\mathbf{M}_1(t) = (\sin(t), \cos(t), t)$ and $\mathbf{M}_2(t) = (\cos(t), \sin(t), t)$, while $m$ hidden systems with periodic behavior are defined as $\mathbf{H}_1(t) = (t + 10, \exp(\sin(t)), \exp(-\cos(t)))$ and $\mathbf{H}_2(t) = (t - 10, \log(\cos(t) + 2), \sqrt{\sin(t) + 1})$. The macrosystem function describes the collective behavior influenced by the periodic nature of microsystems and hidden systems, $\mathbf{C}(t) = (X(t), Y(t), Z(t))$. One can then introduce a new function $\mathbf{B}(t)$ that describes the time-dependent interaction between the microsystems and hidden systems, which can only be discovered after observing at least one full cycle: $\mathbf{B}(t) = g(\mathbf{M}(t), \mathbf{H}(t))$.
This can be defined as:

$$B_x(t) = \int_0^t (\sin(\tau) + \cos(\tau) + \exp(\sin(\tau)) + (\tau + 10))\, d\tau$$

$$B_y(t) = \int_0^t (\cos(\tau) + \sin(\tau) + \exp(-\cos(\tau)) + \log(\cos(\tau) + 2))\, d\tau$$

$$B_z(t) = \int_0^t (2\tau + \sqrt{\sin(\tau) + 1} + \exp(-\cos(\tau)))\, d\tau$$

In the initial time frame $[0,\pi]$, the cyclic behavior is not fully visible when microsystems are defined as $\mathbf{M}_1(t) = (\sin(t), \cos(t), t)$ and hidden systems are defined as $\mathbf{H}_1(t) = (t + 10, \exp(\sin(t)), \exp(-\cos(t)))$.
Likewise, the collective behavior function $\mathbf{B}(t)$ is not fully observable:

$$B_x(t) = \int_0^t (\sin(\tau) + \cos(\tau) + \exp(\sin(\tau)) + (\tau + 10))\, d\tau$$

After expanding the time frame to $[0,4\pi]$, the cyclic behavior becomes fully visible as the microsystems and hidden systems can now complete multiple cycles. The collective behavior function $\mathbf{B}(t)$ reveals the full interaction:

$$B_x(t) = \int_0^{4\pi} (\sin(\tau) + \cos(\tau) + \exp(\sin(\tau)) + (\tau + 10))\, d\tau$$

$$B_y(t) = \int_0^{4\pi} (\cos(\tau) + \sin(\tau) + \exp(-\cos(\tau)) + \log(\cos(\tau) + 2))\, d\tau$$

$$B_z(t) = \int_0^{4\pi} (2\tau + \sqrt{\sin(\tau) + 1} + \exp(-\cos(\tau)))\, d\tau$$

To demonstrate the discovery of the collective behavior function, one may note that the initial observations span 0 to $\pi$. The cyclic behaviors of the microsystems and hidden systems are partially observed, and the collective behavior function $\mathbf{B}(t)$ is not yet fully apparent due to incomplete cycles. Once we expand the observation cycle to 0 to $4\pi$, the full cyclic behaviors of the microsystems and hidden systems are observed. The collective behavior function $\mathbf{B}(t)$ becomes fully observable and reveals the interactions between microsystems and hidden systems. Therefore, once again, by expanding the observer's time frame to include multiple cycles, the initially hidden cyclic behaviors and the collective behavior function $\mathbf{B}(t)$ become fully visible. This demonstrates that the interaction between microsystems and hidden systems, leading to the macrosystem behavior, can only be fully understood when sufficient observational time is allowed to reveal the complete cycles.

## Addition of probability to the observer perspective and its influence on capturing emergent phenomena

Thus far, we have seen that in a classical system of equations, expansion of the observer perspective in 3D coordinate space or 4D spacetime, can result in the acquisition of new information that allows one to mathematically describe systems that previous appeared unpredictable or emergent. Similarly, it is possible to integrate a probability dimension into the system by adding a probability variable $p$ that influences the x, y, and z (and potentially t) coordinates. This probability variable can be thought of as a measure of the likelihood of the observer detecting certain states of the system. The system's behavior will thus depend not only on time but also on this probability variable. To begin, the probability variable can be denoted by $p$, where $0 \leq p \leq 1$. This variable represents the likelihood of observing certain behaviors in the system. Then, equations representing microsystems and hidden systems are modified to include the probability variable $p$. Microsystems are redefined as $\mathbf{M}_1(t,p) = (\sin(t)\cdot p, \cos(t)\cdot p, t\cdot p)$ and $\mathbf{M}_2(t,p) = (\cos(t)\cdot p, \sin(t)\cdot p, t\cdot p)$, while hidden systems are redefined as $\mathbf{H}_1(t,p) = ((t+10)\cdot p, \exp(\sin(t))\cdot p, \exp(-\cos(t))\cdot p)$ and $\mathbf{H}_2(t,p) = ((t-10)\cdot p, \log(\cos(t)+2)\cdot p, \sqrt{\sin(t)+1}\cdot p)$.

The macrosystem function now includes the probability variable:

$\mathbf{C}(t,p) = (X(t,p), Y(t,p), Z(t,p))$

The collective behavior function $\mathbf{B}(t,p)$ now depends on both time and probability:

$$B_x(t,p) = \int_0^t (\sin(\tau)\cdot p + \cos(\tau)\cdot p + \exp(\sin(\tau))\cdot p + (\tau+10)\cdot p)\, d\tau$$

$$B_y(t,p) = \int_0^t (\cos(\tau)\cdot p + \sin(\tau)\cdot p + \exp(-\cos(\tau))\cdot p + \log(\cos(\tau)+2)\cdot p)\, d\tau$$

$$B_z(t,p) = \int_0^t (2\tau\cdot p + \sqrt{\sin(\tau)+1}\cdot p + \exp(-\cos(\tau))\cdot p)\, d\tau$$

The observer's frame of reference can now be expanded to include the probability dimension. Initially, the observer might have a limited range of $p$ values, which can be expanded over time. Let the initial bounds for probability be $p_{\min} = 0$ and $p_{\max} = 0.5$:

$$\mathbf{R}(t,p) = \{(x,y,z,t,p) \mid -5 \leq x \leq 5, -2 \leq y \leq 2, 0 \leq z \leq 5, 0 \leq t \leq \pi, 0 \leq p \leq 0.5\}$$

To include the full range of probability values and expand the observer's frame of reference in probability space:

$$\mathbf{R}'(t,p) = \{(x,y,z,t,p) \mid -15 \leq x \leq 30, -10 \leq y \leq 10, -5 \leq z \leq 30, 0 \leq t \leq 4\pi, 0 \leq p \leq 1\}$$

Therefore, by integrating the probability variable $p$, the system's x, y, and z coordinates in 4D space are now influenced by both time and probability. The observer's frame of reference can initially cover a limited range of probability values, hiding certain behaviors until the frame of reference is expanded to include the full probability range.

To summarize, the initial conditions were, for microsystems, $\mathbf{M}_1(t,p) = (\sin(t)\cdot p, \cos(t)\cdot p, t\cdot p)$, $\quad \mathbf{M}_2(t,p) = (\cos(t)\cdot p, \sin(t)\cdot p, t\cdot p)$, for hidden systems, $\mathbf{H}_1(t,p) = ((t+10)\cdot p, \exp(\sin(t))\cdot p, \exp(-\cos(t))\cdot p)$, $\quad \mathbf{H}_2(t,p) = ((t-10)\cdot p, \log(\cos(t)+2)\cdot p, \sqrt{\sin(t)+1}\cdot p)$ and for the macrosystem,

$$\mathbf{C}(t,p) = \left(\sin(t)\cdot p + \cos(t)\cdot p + \exp(\sin(t))\cdot p + (t+10)\cdot p, \cos(t)\cdot p + \sin(t)\cdot p + \exp(-\cos(t))\cdot p + \log(\cos(t)+2)\cdot p, 2t\cdot p + \sqrt{\sin(t)+1}\cdot p + \exp(-\cos(t))\cdot p\right)$$

The collective behavior function is defined as:

$$B_x(t,p) = \int_0^t (\sin(\tau)\cdot p + \cos(\tau)\cdot p + \exp(\sin(\tau))\cdot p + (\tau+10)\cdot p)\, d\tau$$

$$B_y(t,p) = \int_0^t (\cos(\tau)\cdot p + \sin(\tau)\cdot p + \exp(-\cos(\tau))\cdot p + \log(\cos(\tau)+2)\cdot p)\, d\tau$$

$$B_z(t,p) = \int_0^t (2\tau\cdot p + \sqrt{\sin(\tau)+1}\cdot p + \exp(-\cos(\tau))\cdot p)\, d\tau$$

The observer's frame of reference is then expanded to $\mathbf{R}'(t,p) = \{(x,y,z,t,p) \mid -15 \le x \le 30, -10 \le y \le 10, -5 \le z \le 30, 0 \le t \le 4\pi, 0 \le p \le 1\}$.

This framework shows how the addition of a probability variable can hide or reveal emergent behaviors depending on the observer's probability range, adding another layer to the understanding of emergent systems.

## The relativity of chaotic emergent systems

We started with a simpler, classical system of equations that function as individual components with different behaviors than a collective macrosystem, and are thus initially unable to describe it. Upon expanding the observer perspective in the space of 3D coordinates, time, and/or probability, additional information can be uncovered which allows for a complete description of the collective system, rendering its 'emergent' behavior relative based on observer perspective. This can then be expanded into a simple chaotic system, as emergence is sometimes defined in academic settings as 'high-energy, far-from-equilibrium systems that unpredictably self-organize to reduce chaotic energy flow'. To again demonstrate the relativity of emergent phenomena based on observer perspective, our simple chaotic system will mathematically take the form of a logistic map and surrounding 'hidden systems', that when revealed as the observer changes perspective in space, time, or probability, allow for a complete mathematical description of the previously unpredictable emergent system.

To start, one can again first expand the user perspective in 3D coordinate space by 1) incorporating a logistic map system with a decreasing *r* value, 2) adding a hidden system moving toward the logistic map system, 3) expanding the observer's frame of reference. First, one can define a logistic map with a decreasing *r* value as $x_{n+1} = r_n \cdot x_n \cdot (1 - x_n)$, where $r_n$ is a time-dependent function that decreases after the hidden system reaches a threshold distance:

$$r_n = r_{\max} - \Delta r \cdot n \quad \text{for} \quad n \geq n_{\text{threshold}}$$

Then, the hidden system is defined as a time-dependent system located in 3D space, specifically, $\mathbf{H}(t) = (x_H(t), y_H(t), z_H(t))$. The hidden system starts at an initial distance from the logistic map system's location (assume the logistic map is at $\mathbf{L} = (x_L, y_L, z_L)$) and moves toward it over time as $\mathbf{H}(t) = (x_H(0) - v_x \cdot t, y_H(0) - v_y \cdot t, z_H(0) - v_z \cdot t)$, where $v_x$, $v_y$, and $v_z$ are the velocities in the x, y, and z directions. The hidden system stops when it reaches a threshold distance $d_{\text{threshold}}$ from the logistic map system:

$$\text{Distance}(\mathbf{H}(t), \mathbf{L}) = \sqrt{(x_H(t) - x_L)^2 + (y_H(t) - y_L)^2 + (z_H(t) - z_L)^2} \leq d_{\text{threshold}}$$

Initially, the observer's frame of reference might be too small to include the hidden system:

$$\mathbf{R} = \{(x, y, z) \mid x_{\min} \leq x \leq x_{\max}, y_{\min} \leq y \leq y_{\max}, z_{\min} \leq z \leq z_{\max}\}$$

Which can be defined as $\mathbf{R}_{\text{initial}} = \{(x, y, z) \mid -1 \leq x \leq 1, -1 \leq y \leq 1, -1 \leq z \leq 1\}$.
This initial frame does not include the hidden system:

$$\mathbf{H}(0) = (x_H(0), y_H(0), z_H(0)) \quad \text{with} \quad |x_H(0)| > 1 \text{ or } |y_H(0)| > 1 \text{ or } |z_H(0)| > 1$$

However, as time progresses, the observer can expand their frame of reference to:

$$\mathbf{R}' = \{(x, y, z) \mid x'_{\min} \leq x \leq x'_{\max}, y'_{\min} \leq y \leq y'_{\max}, z'_{\min} \leq z \leq z'_{\max}\}$$

This can be defined as $\mathbf{R}'_{\text{expanded}} = \{(x, y, z) \mid -5 \leq x \leq 5, -5 \leq y \leq 5, -5 \leq z \leq 5\}$. Now, the hidden system is within the observer's frame of reference once it moves closer to the logistic map system.

Therefore, the hidden system is initially outside the observer's view, so the observer cannot see the influence of this system on the logistic map. The hidden system moves toward the logistic map, and at a certain threshold distance, the $r$ value of the logistic map begins to decrease, transitioning the system from chaos to order. Then, the observer expands their frame of reference, eventually including the hidden system and observing the transition from chaos to order. This demonstrates how hidden systems can influence chaotic systems when they enter the observer's frame of reference, leading to emergent order.

To add further depth on how the interaction between the hidden system and the logistic map system leads to a decrease in the $r$ value and a transition from chaotic to periodic behavior, one can start be defining the approach of the hidden system. The hidden system $\mathbf{H}(t)$ moves closer to the logistic map system's coordinates $\mathbf{L}$. Its coordinates are given by:

$$\mathbf{H}(t) = (x_H(0) - v_x \cdot t, y_H(0) - v_y \cdot t, z_H(0) - v_z \cdot t)$$

The distance between the hidden system and the logistic map system at any time $t$ is:

$$d(t) = \sqrt{(x_H(t) - x_L)^2 + (y_H(t) - y_L)^2 + (z_H(t) - z_L)^2}$$

The hidden system continues to move until the distance $d(t)$ reaches the threshold distance $d_{\text{threshold}}$. Once the distance $d(t) \leq d_{\text{threshold}}$, the hidden system stops moving: $\mathbf{H}(t_{\text{stop}}) = \mathbf{H}(t_{\text{threshold}})$. The threshold distance is defined as the point where the hidden system is close enough to influence the logistic map system. When the hidden system stops at $d_{\text{threshold}}$, the interaction with the logistic map system begins. The $r$ value of the logistic map starts to decrease from its initial value $r_{\max}$: $r_n = r_{\max} - \Delta r \cdot (n - n_{\text{threshold}})$. Here,

$n_{\text{threshold}}$ is the iteration step when the hidden system reaches $d_{\text{threshold}}$. The decrease in $r$ continues as $n$ increases, leading to a transition from chaotic to periodic behavior in the logistic map. As the $r$ value decreases, the logistic map transitions from chaotic behavior (where $r_{\max}$ might be in a chaotic regime) to periodic behavior (as $r$ moves into a range that produces periodic or stable fixed points). The system's behavior becomes more ordered, with lower entropy, as $r$ continues to decrease.

In summary, in the initial phase, the hidden system $\mathbf{H}(t)$ moves toward the logistic map system's location. The logistic map remains chaotic as $r$ is constant at $r_{\max}$.
The threshold interaction occurs when the hidden system reaches $d_{\text{threshold}}$, stops, and the $r$ value of the logistic map begins to decrease. Then, the decreasing $r$ value gradually shifts the logistic map from chaotic to periodic behavior during the transition phase. Lastly, the logistic map system becomes periodic, with stable or repeating values of $x_n$ as $r$ decreases sufficiently as it reaches its final phase. Such an interaction demonstrates how a hidden system, once brought into proximity with the primary system (logistic map), can influence it to transition from chaos to order. The observer's frame of reference, initially too small to include the hidden system, must be expanded to see this influence and the resulting emergent behavior.

## Expanding temporal observation period to observe emergent phenomena in chaotic systems

As with the simpler classical system, it is also possible to demonstrate that a particular observation period is required to see the emergence of periodic, orderly behavior in the logistic map system. To do so, a time variable will be introduced into the observer's frame of reference. This time variable will be tied to the period during which the hidden system moves into proximity with the logistic map system, triggering the decrease in the $r$ value. Initially, the hidden system moves toward the logistic map's coordinates over time $t$ and stops at the threshold distance $d_{\text{threshold}}$, $\mathbf{H}(t) = (x_H(0) - v_x \cdot t, y_H(0) - v_y \cdot t, z_H(0) - v_z \cdot t)$, where $v_x$, $v_y$, and $v_z$ are the velocities in the x, y, and z directions, respectively, $t_{\text{threshold}}$ is the time it takes for the hidden system to reach the threshold distance $d_{\text{threshold}}$, at which point the $r$ value starts to decrease. The logistic map equation remains $x_{n+1} = r_n \cdot x_n \cdot (1 - x_n)$, where over time, the $r$ value changes at $t_{\text{threshold}}$:

$$r_n = \begin{cases} r_{\max}, & \text{if } t < t_{\text{threshold}} \\ r_{\max} - \Delta r \cdot (n - n_{\text{threshold}}), & \text{if } t \geq t_{\text{threshold}} \end{cases}$$

The observer's frame of reference now includes a time variable:

$$\mathbf{R}(t) = \{(x, y, z, t) \mid x_{\min} \leq x \leq x_{\max}, y_{\min} \leq y \leq y_{\max}, z_{\min} \leq z \leq z_{\max}, t_{\min} \leq t \leq t_{\max}\}$$

To witness the emergence of order from the chaotic system, the observer must include the time period $[0, t_{\text{threshold}} + \Delta t]$ in their observation. $\Delta t$ is the time required to observe the transition to periodic behavior after the hidden system has influenced the logistic map.

If the observer's time frame is too short, say $t_{\max} < t_{\text{threshold}}$, they will only see the chaotic behavior of the logistic map with $r_n = r_{\text{max}}$. The hidden system will not have reached the logistic map, and no decrease in $r$ will have occurred:\

$$\mathbf{R}_{\text{initial}}(t) = \{(x, y, z, t) \mid -1 \leq x \leq 1, -1 \leq y \leq 1, -1 \leq z \leq 1, 0 \leq t \leq t_{\text{initial}}\}$$

Therefore, to observe the transition to order, the observer must expand their time frame to include $t_{\text{threshold}}$:

$$\mathbf{R}'(t) = \{(x, y, z, t) \mid -5 \leq x \leq 5, -5 \leq y \leq 5, -5 \leq z \leq 5, 0 \leq t \leq t_{\text{threshold}} + \Delta t\}$$

In this expanded time frame, the observer will first see the initial chaotic phase, up to: $t = t_{\text{threshold}}$, where $r_n = r_{\text{max}}$. After $t = t_{\text{threshold}}$, as $r_n$ starts to decrease, the logistic map transitions to periodic behavior.

Therefore, by adding a time variable to the observer's perspective, it can be demonstrated that a sufficient observation period is required to see the emergence of periodic, orderly behavior in the logistic map system. If the observer's time frame is too short, they will only observe chaotic behavior. Once the hidden system moves into proximity and influences the logistic map, and if the observer's time frame is extended to include this period, they will observe the transition from chaos to order.

## Addition of probability to the observer perspective to capture emergent phenomena in chaotic systems

Next, as with the classical system, yet another dimension in the form of probability can be incorporated into the observer's perspective by modifying the equations so that the behavior of the hidden system, including its influence on the logistic map, depends not only on spatial coordinates and time but also on a probability variable $p$. This probability variable can represent the likelihood of the observer detecting certain behaviors or interactions in the system. The hidden system is now described as a function of time $t$, spatial coordinates $(x, y, z)$, and a probability variable $p$, where where $0 \leq p \leq 1$:

$$\mathbf{H}(t, p) = (x_H(0) - v_x \cdot t \cdot p, y_H(0) - v_y \cdot t \cdot p, z_H(0) - v_z \cdot t \cdot p)$$

The hidden system moves toward the logistic map system, but the speed and proximity are influenced by $p$. The hidden system stops moving when it reaches the threshold distance $d_{\text{threshold}}$, which is now also influenced by $p$:

$$d(t, p) = \sqrt{(x_H(t, p) - x_L)^2 + (y_H(t, p) - y_L)^2 + (z_H(t, p) - z_L)^2} \leq d_{\text{threshold}}(p)$$

The logistic map is still given by $x_{n+1} = r_n \cdot x_n \cdot (1 - x_n)$, but now the $r$ value depends on both time and probability:

$$r_n = \begin{cases} r_{\max}, & \text{if } d(t,p) > d_{\text{threshold}}(p) \\ r_{\max} - \Delta r \cdot (n - n_{\text{threshold}}(p)), & \text{if } d(t,p) \le d_{\text{threshold}}(p) \end{cases}$$

The hidden system needs to reach a certain distance $d_{\text{threshold}}(p)$ to trigger the decrease in $r$, which now depends on both $t$ and $p$. The observer's frame of reference is then expanded to include the probability dimension:

$$\mathbf{R}(t,p) = \{(x,y,z,t,p) \mid x_{\min} \le x \le x_{\max}, y_{\min} \le y \le y_{\max}, z_{\min} \le z \le z_{\max}, t_{\min} \le t \le t_{\max}, p_{\min} \le p \le p_{\max}\}$$

Initially, the observer may only be able to observe a limited range of probability values:

$$\mathbf{R}_{\text{initial}}(t,p) = \{(x,y,z,t,p) \mid -1 \le x \le 1, -1 \le y \le 1, -1 \le z \le 1, 0 \le t \le t_{\text{initial}}, 0 \le p \le p_{\text{initial}}\}$$

With this limited range, the hidden system's influence might not be fully observed, and the $r$ value decrease might be hidden.

To observe the full transition to order, the observer needs to expand the probability range:

$$\mathbf{R}'(t,p) = \{(x,y,z,t,p) \mid -5 \le x \le 5, -5 \le y \le 5, -5 \le z \le 5, 0 \le t \le t_{\text{threshold}} + \Delta t, 0 \le p \le 1\}$$

In this expanded frame of reference, the observer will first see chaotic Behavior up to the point where $d(t,p) \le d_{\text{threshold}}(p)$, with $r_n = r_{\max}$, followed by the transition to order, as after the hidden system reaches the threshold influenced by $p$, the $r$ value begins to decrease, leading to periodic behavior in the logistic map.

Therefore, by adding a probability dimension to the observer's perspective, the visibility of the hidden system’s influence on the logistic map is further controlled. Initially, with a limited probability range, certain behaviors (like the decrease in $r$) are hidden. As the observer's probability range is expanded, these behaviors become observable, showing the full transition from chaos to order in the system.

This demonstrates how the emergence of order in a chaotic system can depend on both the time and probability dimensions of the observer's frame of reference. The hidden system's movement and influence on the logistic map are both time-dependent and probability-dependent, requiring sufficient observation in both dimensions to fully capture the emergent behavior.

## A general model of multi-scaled emergent phenomena via Fourier series

Emergent phenomena inherently involve complex interactions across multiple scales of space, time and/or probability that feed back onto one another. Therefore, an adequate general model of emergent phenomena must involve a mathematical superstructure in coordinate space that is capable of incorporating both linear and nonlinear dynamics, and of which can be composed of hierarchies of nested equations. The Fourier series allows an infinite number of oscillators of differing magnitudes to be nested within one another, and therefore act as functions of one another, whether or not all are simultaneously visible, and has been previous used for modeling multiscaled phenomena (Sun and Zhang 2022). This aligns well towards the creation of a general mathematical model of emergent phenomena, because most energetic systems across all space and time scales exhibit behavioral oscillations of one form or another. Likewise, both linear and chaotic behaviors can be integrated into the Fourier series, and therefore, this provides a rich framework for expanding the observer perspective in space, time or probability to reveal hidden dynamics and grant predictability to initially unexplainable systems, where hidden systems can directly be incorporated as elements of the series itself.

A simple, initial Fourier series that begins with a larger, singular oscillator, and then incorporates exponentially larger populations of smaller oscillators at each new level of the system, can be defined as

$$F(t) = \sin(\omega_1 t + \phi_1) + \sum_{k=0}^{2^1-1} \sin(\omega_2 t + \phi_{2,k}) + \sum_{k=0}^{2^2-1} \sin(\omega_3 t + \phi_{3,k}) + \sum_{k=0}^{2^3-1} \sin(\omega_4 t + \phi_{4,k}) + \dots$$

Where $\sin(\omega_n t + \phi_n)$ represents the base function for the n-th term, $\sum_{k=0}^{2^{n-1}-1} \sin(\omega_n t + \phi_{n,k})$ represents the sum of the base function and its $2^{n-1} - 1$ duplicates for the n-th term, each with a unique phase shift $\phi_{n,k}$. An expansion of the equation may therefore take the form of:

$$F(t) = \sin(\omega_1 t + \phi_1) + (\sin(\omega_2 t + \phi_2) + \sin(\omega_2 t + \phi_{2,1})) + (\sin(\omega_3 t + \phi_3) + \sin(\omega_3 t + \phi_{3,1}) + \sin(\omega_3 t + \phi_{3,2}) + \sin(\omega_3 t + \phi_{3,3})) + \dots$$

This model can be modified in a variety of ways to reflect behaviors in the natural world. For example, if we assume the first equation of our series reflects the population of the earth (1) and its oscillation frequency around the sun ($\sim 2\pi \times 1.1574 \times 10^{-5}$ Hz), and the final equation in the series approximates the population of atoms within the earth (~10^50) with vibrational frequencies of roughly ~10^13 Hz, the general equation can be modified in a manner such as:

$$F(t) = e^{\lambda \cdot 1} \sin\left(2\pi \times 1.1574 \times 10^{-5} t + \phi_1\right) + \sum_{k=0}^{e^{\gamma \cdot 0}-1} e^{\lambda \cdot 2} \sin\left(S_2 \cdot \omega_1 t + \phi_{2,k}\right) + \sum_{k=0}^{e^{\gamma \cdot 1}-1} e^{\lambda \cdot 3} \sin\left(S_3 \cdot \omega_1 t + \phi_{3,k}\right) + \sum_{k=0}^{e^{\gamma \cdot 2}-1} e^{\lambda \cdot 4} \sin\left(S_4 \cdot \omega_1 t + \phi_{4,k}\right) + \dots + \sum_{k=0}^{e^{\gamma \cdot 4}-1} e^{\lambda \cdot 5} \sin\left(S_5 \cdot \omega_1 t + \phi_{5,k}\right)$$

Where the starting frequency matching the earth's rotation around the sun begins at $2\pi \times 1.1574 \times 10^{-5}$ Hz, and for subsequent terms, $S_n = e^{\delta(n-1)}$ scales the frequency from the base frequency $\omega_1 = 2\pi \times 1.1574 \times 10^{-5}$ Hz, the number of duplicates for each term is scaled by $e^{\delta(n-1)}$, where $\delta \approx 10.23$, resulting in a frequency of ~10^13 for the final population of terms. Likewise, to ensure the population begins at 1, representing the earth, and ends at a population of ~10^50, representing the order of magnitude of atoms preset in the earth, the population cap of each subsequent term in the sum is multiplied by $e^{\gamma(n-1)}$, where $e^{\gamma \cdot 4} = 10^{50}$ and $\gamma \approx 28.8539$. These exponential multiplies can be further modified and scaled depending on the number of desired, distinct population types within the equation series.

## Expanding 3-dimensional frames of reference to observe emergent phenomena in the Fourier model

For each term in the series, one can define a 3D vector function where each component (x, y, z) is influenced by sine functions with different phases, frequencies, and amplitudes, with the general form:

$$\mathbf{F}(t) = \sum_{n=1}^{\infty} \left( \mathbf{F}_n(t) + \sum_{k=0}^{2^{n-1}-1} \mathbf{F}_{n,k}(t) \right)$$

Where $\mathbf{F}_n(t)$ is the base function for the n-th term in 3D, and $\mathbf{F}_{n,k}(t)$ represents the k-th duplicate of the n-th term with a unique phase shift. Expanding this into x, y, and z components, first for the base function of the n-th term:

$$\mathbf{F}_n(t) = \begin{pmatrix} A_{n,x} \sin(\omega_n t + \phi_{n,x}) \\ A_{n,y} \sin(\omega_n t + \phi_{n,y}) \\ A_{n,z} \sin(\omega_n t + \phi_{n,z}) \end{pmatrix}$$

For the duplicates:

$$\mathbf{F}_{n,k}(t) = \begin{pmatrix} A_{n,x} \sin(\omega_n t + \phi_{n,x,k}) \\ A_{n,y} \sin(\omega_n t + \phi_{n,y,k}) \\ A_{n,z} \sin(\omega_n t + \phi_{n,z,k}) \end{pmatrix}$$

Here, $A_{n,x}, A_{n,y}, A_{n,z}$

are the amplitudes for each dimension, and $\phi_{n,x}, \phi_{n,y}, \phi_{n,z}$ are the phases for each dimension of the base function. For the duplicates, $\phi_{n,x,k}, \phi_{n,y,k}, \phi_{n,z,k}$ represent unique phase shifts for each dimension and duplicate.

To demonstrate the potential for a change in observer perspective to hide or reveal subsequent elements of the Fourier series, the position of the Fourier series point can be represented as $\mathbf{F}(t) = (F_x(t), F_y(t), F_z(t))$, the observer's position as $\mathbf{O} = (O_x, O_y, O_z)$, and the direction vector of the observer's line of sight as $\mathbf{D} = (D_x, D_y, D_z)$. From here, the vector from the observer to the point can be calculated as

$$\mathbf{V} = \mathbf{F}(t) - \mathbf{O} = (F_x(t) - O_x, F_y(t) - O_y, F_z(t) - O_z)$$

with the dot product between **V** and **D** calculated as

$$\mathbf{V} \cdot \mathbf{D} = (F_x(t) - O_x)D_x + (F_y(t) - O_y)D_y + (F_z(t) - O_z)D_z$$

The magnitudes become

$$|\mathbf{V}| = \sqrt{(F_x(t) - O_x)^2 + (F_y(t) - O_y)^2 + (F_z(t) - O_z)^2}\ ,\ |\mathbf{D}| = \sqrt{D_x^2 + D_y^2 + D_z^2}$$

with the cosine of the angle angle $\theta$ between **V** and **D** being

$$\cos(\theta) = \frac{(F_x(t) - O_x)D_x + (F_y(t) - O_y)D_y + (F_z(t) - O_z)D_z}{\sqrt{(F_x(t) - O_x)^2 + (F_y(t) - O_y)^2 + (F_z(t) - O_z)^2} \cdot \sqrt{D_x^2 + D_y^2 + D_z^2}}$$

This can now be incorporated into the visibility function as

$$V_{\text{initial}} = \frac{1}{1 + e^{-\kappa\left(\cos(\theta) - (\theta_{\text{base}} - \alpha \cdot \max(0, k-2))\right)}}$$

Here, $F_x(t), F_y(t), F_z(t)$ represent the x, y, z coordinates of the Fourier series point at time (t), $O_x, O_y, O_z$ are the x, y, z coordinates of the observer's position, $D_x, D_y, D_z$ are the components of the observer's direction vector, (n) is the term number in the series, (k) is the duplicate index within that term, $\theta_{\text{base}}$ is the initial visibility angle threshold, $\alpha$ controls how visibility decreases with each duplicate after the third, and (k) (in the exponent) controls the steepness of the visibility transition. Where $\kappa > 0$ controls steepness, $\theta_{\text{base}} \in [0,\pi]$ is the initial visibility threshold, and $\alpha \geq 0$ adjusts visibility decrease per duplicate after the third. Since $\arccos(\cos(\theta)) = \theta$, the visibility function can be simplified to

$$V(F_x(t), F_y(t), F_z(t), O_x, O_y, O_z, D_x, D_y, D_z, n, k) = \frac{1}{1 + e^{-k\left(\theta - (\theta_{\text{base}} - \alpha \cdot \max(0, k-2))\right)}}$$

Where:

$$\theta = \arccos\left(\frac{(F_x(t) - O_x)D_x + (F_y(t) - O_y)D_y + (F_z(t) - O_z)D_z}{\sqrt{(F_x(t) - O_x)^2 + (F_y(t) - O_y)^2 + (F_z(t) - O_z)^2} \cdot \sqrt{D_x^2 + D_y^2 + D_z^2}}\right)$$

This creates a visibility value that smoothly transitions based on the angle between the direction to the Fourier Series point and the observer's line of sight, adjusted by the duplicate number in the series. Expanding the observer's perspective can involve increasing $\theta_{\text{base}}$ or decreasing the steepness parameter (k) to make more of the series visible. Therefore, this allows different equations in the Fourier series to be treated as 'hidden systems' that can be revealed in space, as well as time.

## Addition of time to the observer perspective and its influence on capturing emergent phenomena in the Fourier model

To demonstrate that changes in the observer's perspective in the time dimension can also hide or reveal new properties of the model, consider a simplified version of the exponentially duplicating Fourier series, with only a single duplication (with a single origin wave and two duplicated waves, otherwise identical except for phase): $F(t) = A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2) + A_2 \sin(\omega_2 t + \phi_2 + \phi_{\text{shift1}})$, where $\phi_{\text{shift1}}$ is the phase shift for the first duplicate. One may imagine the first sine wave as an oscillation representing repetitive behavior of a larger macrosystem, and the two smaller sine waves representing oscillatory behaviors of smaller microsystems within the macrosystem.

A new visibility function $V(t_{\text{obs}}, t, n, k)$ can be created, where $t_{\text{obs}}$ is the observer's time frame, (t) is the time coordinate of the series, (n) represents the term number, and (k) represents the duplicate index. This can be set to depend on the ratio of the observer's time frame to the period of the oscillation, as:

$$V(t_{\text{obs}}, t, n, k) = \frac{1}{1 + e^{-k(\frac{t_{\text{obs}}}{T_n} - \theta_{\text{base}} + \alpha \cdot \max(0, k-1))}}$$

Here, $T_n$ is the period of the n-th term ( $T_n = \frac{2\pi}{\omega_n}$ ), $\theta_{\text{base}}$ is a base threshold that determines when visibility begins to increase, $\alpha$ controls how visibility changes with each duplicate (subsequent wave equations), and (k) in the exponent controls the steepness of the transition from invisible to visible. Adding this to the previous series as an observed series $F_{\text{obs}}(t, t_{\text{obs}})$ results in:

$$F_{\text{obs}}(t, t_{\text{obs}}) = V(t_{\text{obs}}, t, 1, 0) \cdot A_1 \sin(\omega_1 t + \phi_1) + V(t_{\text{obs}}, t, 2, 0) \cdot A_2 \sin(\omega_2 t + \phi_2) + V(t_{.obs}, t, 2, 1) \cdot A_2 \sin(\omega_2 t + \phi_2 + \phi_{\text{shift}1})$$

When $t_{\text{obs}}$ is small relative to $T_1$, the visibility function $V(t_{\text{obs}}, t, 1, 0)$ will be close to 0, making the first sine wave less visible or invisible. At this time frame, the observer would primarily see the oscillations from the second sine wave and its phase-changed version. As $t_{\text{obs}}$ increases, $V(t_{\text{obs}}, t, 1, 0)$ increases, making the first sine wave more visible. The visibility of the second sine wave and its duplicate also depend on $t_{\text{obs}}$ relative to $T_2$, but since only 1-2 rounds of duplication are considered, this may be high enough to be visible depending on the values used.

The generalized visibility function to the n-th term and its k-th duplicate,

$$V(t_{\text{obs}}, t, n, k) = \frac{1}{1 + e^{-k(\frac{t_{\text{obs}}}{T_n} - \theta_{\text{base}} + \alpha \cdot \max(0, k-1))}},$$

holds the general rule that the visibility of each subsequent sine wave component should decrease as the frequency decreases (or period increases) relative to the observer's time frame $t_{\text{obs}}$. Therefore, components with shorter periods (or higher frequencies) become visible sooner, as $t_{\text{obs}}$ increases. Here, $T_n$ is the period of the n-th term in the series. As (n) increases, $T_n$ decreases, because for our emergent Fourier model to represent oscillator hierarchies of multiple system scales, higher-order terms must represent higher frequency components, which would correspond to smaller and faster system behaviors in the physical world, as if moving from planetary motion, to that of populations of humans, to populations of cells, to populations of molecules, to populations of atoms, and so on.

For each term (n) in the series, a cutoff point exists in $t_{\text{obs}}$ where it becomes visible, and is influenced by:

1) Period $T_n$: Smaller $T_n$ (higher frequency) means visibility occurs at smaller $t_{\text{obs}}$, which can be thought of as a rapid observation only being able to capture faster system behaviors.
2) Base Threshold $\theta_{\text{base}}$, which sets the initial visibility threshold.
3) Visibility Change Rate $\alpha$, which controls how visibility changes with each duplicate, with higher values making duplicates less visible unless $t_{\text{obs}}$ is sufficiently large.
4) Duplicate Index (k): Higher (k) values (later duplicates) will have lower visibility unless $t_{\text{obs}}$ is large enough to overcome the phase shift and frequency effects.

Therefore, in our initial, simplified series $F(t) = A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2) + A_2 \sin(\omega_2 t + \phi_2 + \phi_{\text{shift}1})$, the two oscillatory motions comprising $A_2$ would occur at a faster frequency and be observed before the slower $A_1$, and so on for all additional equations added to the series in this manner, where the fastest oscillations are observed before slower, prior ones in the series, demonstrating that the observer perspective in time can once again influence whether a sufficient amount of information is or is not captured towards fully understanding a multiscaled system.

## Addition of probability to the observer perspective and its influence on capturing emergent phenomena in the Fourier model

As seen in natural systems, the probability of successfully observing a fine-scale behavior can decrease as the space and time magnitude of the system behaviors under consideration decrease, as with the 100% probability of observing a full rotation of the earth around the sun, with more difficulty in observing a single atomic vibration. As demonstrated previously, the probability of a successful observation can determine whether or not the observer gains enough information about the system under consideration to fully describe it. Therefore, as the equation population in the modified Fourier series continues to duplicate and increase in number and in frequency for subsequent terms, so too can be the probability of observation be added to modify (in this case, decrease) the chance of successfully observing smaller oscillatory behaviors. This can be achieved through the creation of an exponentially decaying probability function, where the probability of observation decreases by a factor $\beta$ for each step in the series — either by increasing (n) or adding a duplicate (k):

$$P(n, k) = (1 - \beta)^{n-1} \cdot (1 - \beta)^k$$

Here, (n) is the term number in the series, (k) is the duplicate index within that term, and $\beta$ is a parameter between 0 and 1 that controls the rate of decrease in probability, with a higher $\beta$ indicating a faster decrease in visibility probability with each new term or duplicate. From our prior equations, if we assume we have reached the $t_{\text{obs}}$ threshold where all frequencies, including the first frequency, are visible time-wise, we now add the probabilistic aspect to determine the final visibility. The visibility function from time time $V(t_{\text{obs}}, t, n, k)$ will now be multiplied by the probability function (P(n, k)) to get the effective visibility:

$$V_{\text{eff}}(t_{\text{obs}}, t, n, k) = V(t_{\text{obs}}, t, n, k) \cdot P(n, k)$$

For the base function that we can consider as sineWaveA ($n = 1, k = 0$),

$$V_{\text{eff}}(t_{\text{obs}}, t, 1, 0) \cdot \text{sineWaveA}(t) = V(t_{\text{obs}}, t, 1, 0) \cdot 1 \cdot A_1 \sin(\omega_1 t + \phi_1),$$

since $P(1,0) = 1$, the probability of observing the base function is 100%. For the second term with its duplicate, sineWaveB ($n = 2, k = 0$),

$$V_{\text{eff}}(t_{\text{obs}}, t, 2, 0) \cdot \text{sineWaveB}(t) = V(t_{\text{obs}}, t, 2, 0) \cdot (1 - \beta) \cdot A_2 \sin(\omega_2 t + \phi_2),$$

and for its phase-changed duplicate ($n = 2, k = 1$),

$V_{\text{eff}}(t_{\text{obs}}, t, 2, 1) \cdot \text{sineWaveB}_{\text{phaseChange1}}(t) = V(t_{\text{obs}}, t, 2, 1) \cdot (1-\beta)^2 \cdot A_2 \sin(\omega_2 t + \phi_2 + \phi_{\text{shift1}})$

Therefore, for a general term (n) and its (k)-th duplicate, the equation would be:

$$F_{\text{obs}}(t, t_{\text{obs}}, n, k) = V(t_{\text{obs}}, t, n, k) \cdot (1-\beta)^{n-1} \cdot (1-\beta)^k \cdot A_n \sin(\omega_n t + \phi_n + \phi_{n,k})$$

Here, $V(t_{\text{obs}}, t, n, k)$ is the time-based visibility function, which we assume is 1 since $t_{\text{obs}}$ is large enough to observe all frequencies, $(1-\beta)^{n-1} \cdot (1-\beta)^k$ is the probability of observing this specific component, $A_n$ is the amplitude of the n-th term, $\omega_n$ is the frequency of the n-th term, $\phi_n$ is the phase of the base function for the n-th term, and $\phi_{n,k}$ is the additional phase shift for the k-th duplicate of the n-th term. Implementation:

When $\beta$ is small, the decrease in probability with each new term or duplicate is gradual, meaning higher frequency components (later duplicates or terms) still have a good chance of being observed. As $\beta$ increases, the probability drops off more sharply, making it less likely to observe faster frequencies or later duplicates. This ensures that while time $t_{\text{obs}}$ allows for the potential visibility of all components, the probability function (P(n, k)) introduces a selective mechanism where the likelihood of observing each new component decreases, favoring the observation of the initial, slower frequencies over the faster, more recent duplications. This reflects the scenario where the observer's ability to perceive faster oscillations diminishes with each new addition to the series.

## Conclusion

This writing uses several mathematical examples to demonstrate that the addition of information to the observer perspective in space, time or probability coordinates can allow one to explain and predict an otherwise unexplainable or unpredictable system. Therefore, many systems currently deemed emergent may simply be labeled as such due to lack of information in the observer perspective. This eludes to the possibility that an intentional study of emergent phenomena, spanning complex diseases to large scale climate systems, may help better understand and one day, predict them.

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